Mathematics 222B Lecture 22 Notes

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1 Introduction to Calculus of Variations

1.1 Motivation and general setup

Now, we will begin the final part of this course, where we will study nonlinear PDEs. Calculus of variations gives us a lot of extra structure which is helpful in studying nonlinear PDEs. The reference is sections 8.1, 8.6 in Evans' book, but we will give some more focus on the formalism than Evans.

In the calculus of variations, we are looking for the critical points of a functional $F : X \to \mathbb{R}$; these are necessary to find extrema and is motivated by optimization problems. We will give some more motivations later. For us X will be a set of functions, which differentiates this from an ordinary calculus problem.

Example 1.1 (Energy minimizing curves)**.** Given a curve $\gamma : [0,1] \to \mathbb{R}^d$, we can associate the **energy**

$$E[\gamma] = \int_0^1 |\dot{\gamma}(t)|^2 \, dt.$$

What are the minimizers of $\mathbb{E}[\gamma]$?

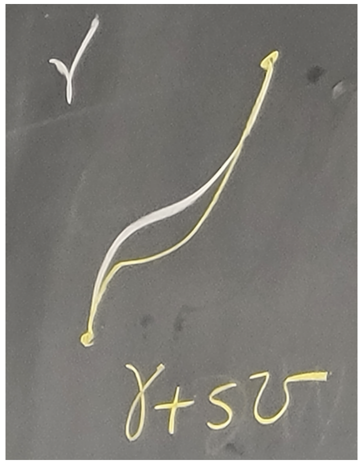
To solve problems like this, we need to generalize what we do in usual calculus: We want to find a way to say something like "$\nabla E[\gamma] = 0$." The idea is to think of directional derivatives instead. We can equivalently find a γ such that

$$\frac{d}{ds} E[\gamma + sv]|_{s=0} = 0$$

for all $v : [0,1] \to \mathbb{R}^d$ in a reasonable class.

1.2 Examples of the Euler-Lagrange equation

For simplicity, we assume $\gamma \in C^\infty([0,1];\mathbb{R}^d)$ and $v \in C_c^\infty((0,1);\mathbb{R}^d) =: \mathscr{A}$.



We can write out

$$\begin{aligned}\frac{d}{ds}E[\gamma+sv]|_{s=0} &= \frac{d}{ds}\int_0^1 \left|\frac{d}{dt}(\gamma(t)+sv(t))\right|^2 dt\bigg|_{s=0}\\ &= 2\int_0^1 \frac{d}{dt}(\gamma+sv)\frac{d}{dt}v\bigg|_{s=0} dt\\ &= 2\int_0^1 \dot\gamma\frac{d}{dt}v\,dt\\ &= -2\int_0^1 \ddot\gamma v\,dt.\end{aligned}$$

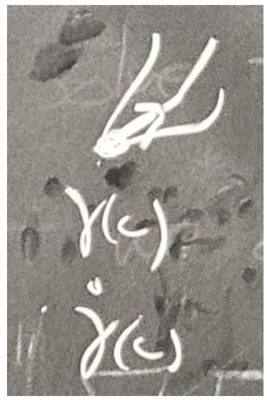
So we see that

$$\begin{aligned}0=\frac{d}{ds}E[\gamma+sv]|_{s=0} \qquad \forall v\in\mathscr{A} &\iff 0=\int_0^1 \ddot\gamma v\,dt \qquad \forall v\in\mathscr{A}\\ &\iff \ddot\gamma = 0 \qquad \text{on } (0,1).\end{aligned}$$

Our critical point condition gave us a differential equation. This is called the **Euler-Lagrange equation**. It tells us that energy minimizing curves are straight lines (geodesics). If we take $F[u]$, where $u : U \to \mathbb{R}^N$ and $U \subseteq \mathbb{R}^n$ with $n \geq 2$, we will in general get a PDE for our critical points. There are two ways to generalize this example:

1. If we look for the minimum of F we need some extra condition, such as the idea of **convexity** of F. This leads to elliptic PDEs. Chapter 8 of Evans' book focuses mostly on this approach.

2. We can interpret this as a Lagrangian mechanics problem. This is when there is a natural time variable in the problem. Here, we do not worry about minimizing F; we just look for critical points. In this setting, critical points give an equation (like $\ddot{\gamma} = 0$) which tells us locally how the curve will evolve given initial conditions.



   This is known as the principle of **stationary action**, in which case, we call F the **action**.

Here are some examples of Euler-Lagrange equations corresponding to various calculus of variation problems.

Example 1.2 (Dirichlet's principle)**.** Let $u : U \to \mathbb{R}$, where U is an open, bounded C^∞ domain contained in $\mathbb{R}^d$. We take $u \in C^\infty(\overline{U})$ and take our functional to be

$$F[u] = \frac{1}{2} \int_U |Du|^2 \, dx.$$

(Compare this with our equation for geodesics). The critical points satisfy the PDE $-\Delta u = 0$.

Example 1.3 (Action principle for the wave equation)**.** Let $u : O \to \mathbb{R}$, where O is an open subset of $\mathbb{R}^{1+d}_{t,x}$. We take $u \in C^\infty(\overline{U})$ and

$$\mathcal{S}[u] = \int_O (\partial_t u)^2 - |Du|^2 \, dt \, dx.$$

The critical points satisfy the wave equation in O.

1.3 First variation (the Euler-Lagrange equation)

From now on, we restrict our attention to functionals of the form

$$F[u] = \int_U L(Du(x), u(x), x) \, dx,$$

where $L : (p, z, x) : \mathbb{R}^d \times \mathbb{R} \times U \to \mathbb{R}$ is called the **Lagrangian density**. For our notation, we will use brackets when we are talking about u as a function as a whole and parentheses when we are talking about values of u.

To think of first variations, we think of directional derivatives. Take $u \in \mathscr{A}$ and variations $v \in \mathscr{A}_0$. Then we will try to form

$$\frac{d}{ds} F[u + sv]|_{s=0}.$$

Remark 1.1. When $\mathscr{A}_0 = \mathscr{A}$, people in functional analysis call this the **Gâteaux derivative**.

In our case, for simplicity, eww assume $\mathscr{A} = C^\infty(\overline{U})$ and $\mathscr{A}_0 = C_c^\infty(U)$. The assumption on $\mathscr{A}_0$ is okay, and the assumption on $\mathscr{A}$ is restrictive but easily removable. We get

$$\begin{aligned}
D_v F[u] &= \frac{d}{ds} F[u + sv]\bigg|_{s=0} \\
&= \frac{d}{ds} \int_U L(D(u + sv), u + sv, x)\, dx\bigg|_{s=0} \\
&= \int_U \frac{d}{ds} L(D(u + sv), u + sv, x)|_{s=0}\, dx \\
&= \int_U \partial_j v \left(\frac{\partial}{\partial_{p_j}} L\right)(Du, u, x) + v(\partial_z L)(Du, u, x)\, dx \\
&= \int_U v \left(-\partial_j \left(\left(\frac{\partial}{\partial p_j} L\right)(Du, u, x)\right) + (\partial_z L)(Du, u, x)\right) dx.
\end{aligned}$$

In particular, if $D_v F[u] = 0$ for all $v \in \mathscr{A}_0$,

$$\left(-\partial_j \left(\frac{\partial}{\partial p_j} L\right) + \partial_z L\right)(Du, u, x) = 0$$

in U. This is the **Euler-Lagrange equation**.

Example 1.4 (Dirichlet's principle)**.** In this example, $L = \frac{1}{2}|p|^2$, so the Euler-Lagrange equation is

$$0 = \partial \underbrace{\left(\frac{\partial}{\partial p_j} \frac{1}{2}|p|^2\right)}_{p_j} |_{p=D_u},$$

which gives us $-\Delta u = 0$.

Example 1.5 (Action principle for the wave equation)**.** In this example, $L = \frac{1}{2}p_0^2 - \frac{1}{2}|p_x|^2$. The Euler-Lagrange equation is

$$0 = -\partial_t \underbrace{\left(\frac{\partial}{\partial p_0} L\right)}_{p_0} |_{p_{t,x}=D_{t,x}u} - \sum_{j=1}^d \partial_j \left(\frac{\partial}{\partial p_j} L\right)\bigg|_{p_{t,x}=D_{t,x}u},$$

so we get $-\partial_t^2 u + \Delta u = 0$.

Remark 1.2. In calculus,

$$D_v F[u] = \langle v, \nabla F[u] \rangle.$$

With a choice of inner product, we can define the **gradient** of F. In our case, we have computed that

$$D_v F[u] = \int_U v(\cdots)\, dx.$$

With respect to the L^2 inner product $\langle \cdot, \cdot \rangle = \int_U uv\, dx$, we have

$$D_v F[u] = \langle v, \text{LHS of E-L equation} \rangle.$$

Because of this, the left hand side of the Euler-Lagrange equation is sometimes called the **L^2-gradient of** F, ∇F. Note that ∇F is now an operator $u \mapsto \nabla F[u]$.

1.4 Second order variation

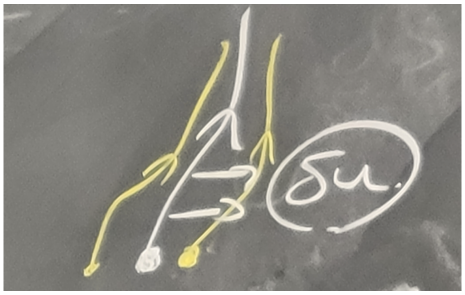
Again, start from directional derivatives. In calculus, the proper way to think about second order directional derivatives is the following:

$$D_{v,w} F[u] = \frac{d}{ds}\frac{d}{dt} F[u + sv + tw]|_{s=0,t=0}.$$

In our case, we define second order directional derivatives of F by this formula. There are two interpretations of the second order variation:

1. In the context of minimization, we can think of this as the Hessian of F at u contracted with two direction vectors v, w. We can then try to come up with a second derivative test to see if a critical point is a maximizer or minimizer.

2. We can think of this as a linearized operator around a critical point. Often, we are not just interested in a single solution but also nearby solutions; this allows us to think about variation through critical points.



In geometry, this is the notion of **Jacobi shifts**. We want $u(x;\lambda)$ such thst $u(x;0) = u(x)$ is a given critical point and $u(x;\lambda)$ are all critical points. We can write this as

$$\nabla F[u(x,\lambda)] = 0,$$

or

$$D_v F[u(x;\lambda)] = 0.$$

We can then differentiate this in λ and get that

$$\frac{d}{d\lambda} \nabla F[u(x,\lambda)]\bigg|_{\lambda=0} = 0,$$

or

$$\frac{d}{d\lambda} D_v F[u(x;\lambda)]\bigg|_{\lambda} = 0, \qquad (v \in \mathscr{A}_v),$$

where

$$u(x;\lambda) = u(x) + \lambda \frac{\partial}{\partial \lambda} u\bigg|_{\lambda=0} = u(x) + \delta u.$$

We can write

$$D_{\delta u} D_v F[u] = 0,$$

which is called the **linearization** of the Euler-Lagrange equation around U for δu.

1.5 Nöther's principle

This principle can be summarized with a slogan: "(continuous) symmetries of the action correspond to conservation laws for solutions." In nonlinear PDEs, conservation laws are very useful but hard to come by. Oftentimes, you have no idea what the solution to an equation is but you know that it's invariant under, say, time translations. This gives you a conserved quantity we can study to understand the solutions to an equation.

Introduce a parameter τ and think about a 1-parameter family of variations.

Definition 1.1. $x \mapsto X(x,\tau)$ si called the **domain variation**, and $u \mapsto u(x,\tau)$ is called the **function variation**.

Example 1.6. We can, for example, take $X(x,\tau) = x - \tau e_1$ and $u(x,\tau) = u(x - \tau e_1)$.

Definition 1.2. F is **invariant** under $X(\cdot,\tau)$ and $a(\cdot,\tau)$ if

$$U(\tau) = X(U,\tau), \qquad u(x,0) = u(x), X(x,0) = x,$$

$$\int_U L(Du(x,\tau), u(x,\tau), x)\, dx = \int_{U(\tau)} L(Du, u, x)\, dx.$$

Theorem 1.1 (Nöther's principle)**.** *In this case,*

$$\partial_j(m\partial_{p_j}L - Lv^j) = m \cdot \left(\partial_j \frac{\partial}{\partial p_j} L - \partial_z L\right)\Big|_{p=Du,z=u},$$

where $m(x) = \frac{\partial}{\partial\tau}u(x,\tau)|_{\tau=0}$ *and* $v^j(x) = \frac{\partial}{\partial\tau}X^j(x,\tau)$.

The key idea is that $\partial_j \frac{\partial}{\partial p_j}L - \partial_z L|_{p=Du,z=u}$ is ∇F. We will discuss this in more detail next time.